International Journal of Theoretical Physics, Vol. 34, No. 10, 1995

Derivation of Quantum Mechanics from the Boltzmann Equation for the Planck Aether

F. Winterberg[1]

Received December 7, 1994

The Planck aether hypothesis assumes that space is densely filled with an equal number of locally interacting positive and negative Planck masses obeying an exactly nonrelativistic law of motion. The Planck masses can be described by a quantum mechanical two-component nonrelativistic operator field equation having the form of a two-component nonlinear Schrödinger equation, with a spectrum of quasiparticles obeying Lorentz invariance as a dynamic symmetry for energies small compared to the Planck energy. We show that quantum mechanics itself can be derived from the Newtonian mechanics of the Planck aether as an approximate solution of Boltzmann's equation for the locally interacting positive and negative Planck masses, and that the validity of the nonrelativistic Schrödinger equation depends on Lorentz invariance as a dynamic symmetry. We also show how the many-body Schrödinger wave function can be factorized into a product of quasiparticles of the Planck aether with separable quantum potentials. Finally, we present a possible explanation of wave function collapse as a kind of enhanced gravitational collapse in the presence of the negative Planck masses.

1. INTRODUCTION

The *Planck aether hypothesis* is the assumption that space is densely filled with an equal number of positive and negative Planck masses, locally interacting with each other, but otherwise not the source of any long-range field. It is furthermore assumed that the Planck masses obey a nonrelativistic law of motion, giving preference to the Galilei group as the more fundamental kinematic symmetry of nature. This hypothesis is in line with Planck's conjecture of 1899 that the only truly universal constants are $\hbar$, G, and c, and that all other constants of physics should be reduced to them.

[1] Desert Research Institute, University and Community College System of Nevada, Reno, Nevada.

0020-7748/95/1000-2145$07.50/0

In the *Planck aether model* the law of motion is specified to be a two-component nonrelativistic Heisenberg-type operator field equation (Winterberg, 1988, 1994)

$$i\hbar \frac{\partial \psi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_p} \nabla^2 \psi_\pm \pm 2\hbar c r_p^2 (\psi_\pm^\dagger \psi_\pm - \psi_\mp^\dagger \psi_\mp)\psi_\pm \tag{1.1}$$

where $\psi_\pm$ are field operators obeying the commutation relations

$$\begin{aligned} [\psi_\pm(\mathbf{r})\psi_\pm^\dagger(\mathbf{r}')] &= \delta(\mathbf{r} - \mathbf{r}') \\ [\psi_\pm(\mathbf{r})\psi_\pm(\mathbf{r}')] &= [\psi_\pm^\dagger(\mathbf{r})\psi_\pm^\dagger(\mathbf{r}')] = 0 \end{aligned} \tag{1.2}$$

In (1.1), $r_p = (\hbar G/c^3)^{1/2}$ and $m_p = (\hbar c/G)^{1/2}$ are the Planck length and mass, respectively. In the Planck aether model each Planck mass occupies the volume r_p^3.

From the Planck aether model one can compute a spectrum of particles which together with their interactions greatly resemble the known spectrum of elementary particles. All these particles are quasiparticles of collective excitations of the Planck aether, and because wavelike disturbances are propagated in the Planck aether with the velocity of light, Lorentz invariance emerges as a derived dynamic symmetry for energies small compared to the Planck energy. It must be stressed that the Planck aether has little resemblance to the pre-Einstein aether models of the late 19th century, where the aether was a substance separate and in addition to ordinary matter. In the Planck aether model, the aether is rather the fundamental field, from which all elementary particles and their interactions would have to be derived, as in Heisenberg's nonlinear spinor theory. However, unlike Heisenberg's proposed fundamental field equation, which is invariant under the noncompact Lorentz group, the field equation of the Planck aether is invariant under the compact Galilei group. Accordingly, the kinds of divergences which occur in Heisenberg's theory are absent from the Planck aether model.

With the exception of the Planck-mass particles, all particles are quasiparticles, but only the quasiparticles obey Lorentz invariance as a dynamic symmetry. The Planck masses, in contrast, are subject to Galilei invariance. It would therefore be much more appealing if the Planck masses would be subject not only to a nonrelativistic law of motion, but to a classical Newtonian law of motion. Because the Planck aether has both positive and negative masses, such a description would require an extension of Newtonian mechanics to negative masses. Following Newton's conjecture that hard, frictionless spheres are the ultimate building blocks of matter, and by identifying Newton's frictionless spheres with the Planck masses, the Planck aether would be solely described by the kinetic energy of the Planck masses with all forces reduced to kinematic boundary conditions at the surface of the spheres. Newton's system of frictionless spheres where the forces are replaced by kinematic boundary conditions at the surface of the spheres is probably the most perfect mechanical counterpart to Einstein's vacuum field equation of the gravitational field, where the forces are eliminated by the metric of a non-Euclidean space-time.

We therefore ask whether, by identifying Newton's hard, frictionless spheres with the Planck masses, but permitting the existence of negative masses, we can derive quantum mechanics from the Planck aether hypothesis, as we have been able to derive special relativity from the same hypothesis. In support of this conjecture, we may note that the fundamental force which can be constructed from $\hbar$, G, and c is

$$F_{\mathrm{p}} = c^4/G \tag{1.3}$$

and does not contain $\hbar$. If everything should be reduced to $\hbar$, G, and c, then $\hbar$ should not enter the force governing Newton's hard, frictionless spheres.

2. NEWTONIAN MECHANICS OF THE PLANCK AETHER

Taking the Hartree approximation of (1.1), by replacing the field operators with their expectation values, one obtains the Schrödinger equation for a positive (or negative) Planck mass

$$i\hbar \frac{\partial \psi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_{\mathrm{p}}} \nabla^2 \psi_\pm + U(\mathbf{r})\psi_\pm \tag{2.1}$$

in the average potential

$$U(r) = 2\hbar c r_{\mathrm{p}}^2 \langle |\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- | \rangle \tag{2.2}$$

generated by all Planck masses. Through its kinetic energy term the one-body Schrödinger equation alone implies the replacement of the classical mechanical momentum by the operator $p = (\hbar/i)\,\partial/\partial q$, obeying the commutation relation $[pq] = \hbar/i$. For a field-theoretic treatment of the many-body problem, as in (1.1), it is thus sufficient to prove that the one-body Schrödinger equation (2.1) can be derived from the Newtonian mechanics of the Planck aether.

Within the Planck aether, a Planck mass is subject to collisions with both positive and negative Planck masses, whereby the average force exerted by all Planck masses on one Planck mass can be described by a potential. The collision between two negative Planck masses has the same outcome as the collision between two positive Planck masses, but the collision of two Planck masses of opposite sign leads to what Schrödinger (1930, 1931) called a "Zitterbewegung" (quivering motion). This can be seen as follows: As is true for the collision of Planck masses of equal sign, the collision of a positive with a negative Planck mass does not change the tangential velocity component. Only the normal component is changed, but the outcome is different. If v'_+ and v'_- are the normal velocity components before, v_+ and v_- those after the collision, energy and momentum conservation imply that

$$v_+^2 - v_-^2 = v_+'^2 - v_-'^2$$

$$v_+ - v_- = v'_+ - v_- \tag{2.3}$$

Rewriting the first of (2.3) as $(v_+ - v_-)(v_+ + v_-) = (v'_+ - v'_-)(v'_+ + v'_-)$ and dividing it by the second, one has

$$v_+ + v_- = v'_+ + v'_- \tag{2.4}$$

and hence

$$v_+ = v'_+$$

$$v_- = v'_- \tag{2.5}$$

It thus follows that the collision between a positive and a negative Planck mass does not change the velocity of the colliding Planck masses, neither in magnitude nor direction, but it permits a spatial parallel displacement of the trajectories. Expressed in terms of Planck's fundamental units, this displacement should be equal to

$$\delta = (1/2)a_p t_p^2 \tag{2.6}$$

where $a_p = F_p/m_p = c^4/Gm_p = c^{7/2}/(\hbar G)^{1/2}$ and $t_p = r_p/c$. One thus finds that

$$\delta = \pm\ (1/2)(\hbar G/c^3)^{1/2} = \pm(1/2)r_p = \pm\hbar/2m_p c \tag{2.7}$$

which is just the radius of the Zitterbewegung derived by Schrödinger from the Dirac equation with the Zitterbewegung velocity $a_p t_p = c$. Whereas the fundamental force $F_p = c^4/G$ does not depend on $\hbar$, the Zitterbewegung displacement certainly does. How this Zitterbewegung should be viewed is illustrated in Fig. 1.

3. DERIVATION OF THE SCHRÖDINGER EQUATION FROM THE BOLTZMANN EQUATION

To obtain the equation of motion of a single Planck mass immersed in the Planck aether one has to solve the Boltzmann equation. We use it in the form (Landau and Lifshitz, 1981)

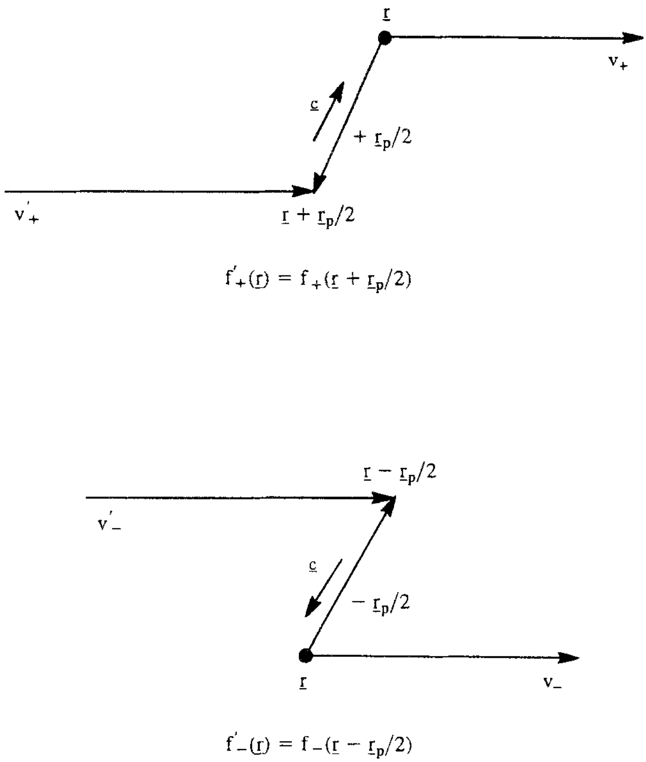


Fig. 1. "Zitterbewegung" displacement $\delta = \pm r_p/2$ of a positive (negative) Planck mass colliding with a negative (positive) Planck mass.

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = \int \mathbf{v}_{\text{rel}}(f'f'_1 - ff_1)\, d\sigma\, d\mathbf{v}_1 \tag{3.1}$$

where f is the distribution function of the colliding particles, f', f'_1 before and f, f_1 after the collision, with f'_1 and f_1 the distribution functions of the particles which by colliding with those belonging to f' and f change the distribution function from f' to f. Furthermore, $\mathbf{v}_{\text{rel}}$ is the magnitude of the relative collision velocity and σ the collision cross section. The particle number density and average velocity are

$$n(\mathbf{r}, t) = \int f(\mathbf{v}, \mathbf{r}, t)\, d\mathbf{v}$$

$$V(\mathbf{r}, t) = \int \mathbf{v} f(\mathbf{v}, \mathbf{r}, t)\, d\mathbf{v}/n(\mathbf{r}, t) \tag{3.2}$$

The acceleration $\mathbf{a}$ is obtained from the force $-\nabla U$ of the potential $U(\mathbf{r})$.

The Boltzmann equation for the positive and negative masses of the Planck aether is

$$\frac{\partial f_\pm}{\partial t} + \mathbf{v}_\pm \cdot \frac{\partial f_\pm}{\partial \mathbf{r}} \mp \frac{1}{m_p} \frac{\partial U}{\partial \mathbf{r}} \cdot \frac{\partial f_\pm}{\partial \mathbf{v}} = 4\alpha c r_p^2 \int (f'_\pm f'_\mp - f_\pm f_\mp)\, d\mathbf{v}_\mp \quad (3.3)$$

where we have set $\sigma = (2r_p)^2 = 4r_p^2$ and $\mathbf{v}_{rel} = \alpha c$, with α a still to be determined dimensionless constant. With the Zitterbewegung equal to the velocity of light suggests that $\alpha = 1$.

According to (2.7) one has (see Fig. 1)

$$f'_\pm(\mathbf{r}) = f_\pm(\mathbf{r} \pm \mathbf{r}_p/2) \quad (3.4)$$

where one has to average over all possible displacements and Zitterbewegung velocities. The direction of the Zitterbewegung motion is in the opposite direction of the displacement $\mathbf{r}_p/2$. With (3.4) the integrand in the collision integral of (3.3) becomes

$$f'_\pm f'_\mp - f_\pm f_\mp = f_\pm\left(\mathbf{r} \pm \frac{\mathbf{r}_p}{2}\right) f_\mp\left(\mathbf{r} \mp \frac{\mathbf{r}_p}{2}\right) - f_\pm(\mathbf{r}) f_\mp(\mathbf{r}) \quad (3.5)$$

Expanding $f_\pm(\mathbf{r} \pm \mathbf{r}_p/2)$ and $f_\mp(\mathbf{r} \mp \mathbf{r}_p/2)$ into a Taylor series

$$\begin{aligned} f_\pm\left(\mathbf{r} \pm \frac{\mathbf{r}_p}{2}\right) &= f_\pm \pm \frac{\mathbf{r}_p}{2} \cdot \frac{\partial f_\pm}{\partial \mathbf{r}} + \frac{\mathbf{r}_p^2}{8} \frac{\partial^2 f_\pm}{\partial \mathbf{r}^2} + \cdots \\ f_\mp\left(\mathbf{r} \mp \frac{\mathbf{r}_p}{2}\right) &= f_\mp \mp \frac{\mathbf{r}_p}{2} \cdot \frac{\partial f_\mp}{\partial \mathbf{r}} + \frac{\mathbf{r}_p^2}{8} \frac{\partial^2 f_\mp}{\partial \mathbf{r}^2} + \cdots \end{aligned} \quad (3.6)$$

one finds up to second order

$$\begin{aligned} f'_\pm f'_\mp - f_\pm f_\mp \simeq {} & \pm\frac{\mathbf{r}_p}{2}\left(f_\mp \frac{\partial f_\pm}{\partial \mathbf{r}} - f_\pm \frac{\partial f_\mp}{\partial \mathbf{r}}\right) \\ & - \frac{\mathbf{r}_p^2}{4} \frac{\partial f_\pm}{\partial \mathbf{r}} \frac{\partial f_\mp}{\partial \mathbf{r}} + \frac{\mathbf{r}_p^2}{8}\left(f_\mp \frac{\partial^2 f_\pm}{\partial \mathbf{r}^2} + f_\pm \frac{\partial^2 f_\mp}{\partial \mathbf{r}^2}\right) \end{aligned} \quad (3.7)$$

with higher order terms suppressed by the Planck length. With good approximation one may set

$$f_\mp(\mathbf{v}_\mp, \mathbf{r}, t) \simeq f_\pm(\mathbf{v}_\pm, \mathbf{r}, t) \quad (3.8)$$

whereby (3.7) becomes

$$f'_{\pm}f'_{\mp} - f_{\pm}f_{\mp} \simeq \frac{\mathbf{r}_p^2}{4}\left(\frac{\partial f_{\pm}}{\partial \mathbf{r}}\right)^2 + \frac{\mathbf{r}_p^2}{4} f_{\pm} \frac{\partial^2 f_{\pm}}{\partial \mathbf{r}^2}$$
$$= \left(\frac{\mathbf{r}_p}{2}\right)^2 f_{\pm}^2 \frac{\partial^2 \log f_{\pm}}{\partial \mathbf{r}^2}$$
$$\simeq \left(\frac{\mathbf{r}_p}{2}\right)^2 f_{\pm} f_{\mp} \frac{\partial^2 \log f_{\pm}}{\partial \mathbf{r}^2} \tag{3.9}$$

To average the Zitterbewegung displacement over a sphere with the volume to surface ratio $(r_p/2)^3/(r_p/2)^2 = r_p/2$, one must apply to (3.9) the operator $(1/2)\mathbf{r}_p \cdot \partial/\partial\mathbf{r}$, and to average over the Zitterbewegung velocity $\mathbf{c}$ (with $\mathbf{c}$ directed opposite to $\mathbf{r}_p$) one must apply in addition the operator $\mathbf{c} \cdot \partial/\partial\mathbf{v}_{\pm}$. Then by integrating (3.3) over $d\mathbf{v}_{\mp}$, where one may set with sufficient accuracy $\int f_{\mp}\, d\mathbf{v}_{\mp} \simeq 1/2r_p^3$, which is the number density of one Planck mass species in the undisturbed Planck aether, one finds

$$\frac{\partial f_{\pm}}{\partial t} + \mathbf{v}_{\pm} \cdot \frac{\partial f_{\pm}}{\partial \mathbf{r}} \mp \frac{1}{m_p} \frac{\partial U}{\partial \mathbf{r}} \cdot \frac{\partial f_{\pm}}{\partial \mathbf{v}_{\pm}}$$
$$= -\frac{\alpha c^2 \mathbf{r}_p^2}{4} \frac{\partial^2}{\partial \mathbf{v}_{\pm} \partial \mathbf{r}} \left(f_{\pm} \frac{\partial^2 \log f_{\pm}}{\partial \mathbf{r}^2} \right) \tag{3.10}$$

To obtain an approximate solution of the Boltzmann equation (3.10) we compute its zeroth and first moments. The zeroth moment is obtained by integrating (3.10) over $d\mathbf{v}_{\pm}$, with the result that

$$\frac{\partial n_{\pm}}{\partial t} + \frac{\partial (n_{\pm}\mathbf{V}_{\pm})}{\partial \mathbf{r}} = 0 \tag{3.11}$$

This is the continuity equation for the macroscopic quantities (3.2). The first moment is obtained by multiplying (3.10) with $\mathbf{v}_{\pm}$ and integrating over $d\mathbf{v}_{\pm}$. For the logarithmic term in (3.10) we approximately set $\partial^2 \log f_{\pm}/\partial\mathbf{r}^2 \simeq \partial^2 \log n_{\pm}/\partial\mathbf{r}^2$. We thus find

$$\frac{\partial (n_{\pm}\mathbf{V}_{\pm})}{\partial t} + \frac{\partial (n_{\pm}\mathbf{V}_{\pm} \cdot \mathbf{V}_{\pm})}{\partial \mathbf{r}} = \mp \frac{n_{\pm}}{m_p} \frac{\partial U}{\partial \mathbf{r}} + \frac{\alpha c^2 r_p^2}{4} \frac{\partial}{\partial \mathbf{r}} \left(n_{\pm} \frac{\partial^2 \log n_{\pm}}{\partial \mathbf{r}^2} \right) \tag{3.12}$$

With the help of (3.11) this can be written as

$$\frac{\partial \mathbf{V}_{\pm}}{\partial t} + \mathbf{V}_{\pm} \cdot \frac{\partial \mathbf{V}_{\pm}}{\partial \mathbf{r}} = \mp \frac{1}{m_p} \frac{\partial U}{\partial \mathbf{r}} + \frac{\alpha \hbar^2}{4 m_p^2} \frac{1}{n_{\pm}} \frac{\partial}{\partial \mathbf{r}} \left(n_{\pm} \frac{\partial^2 \log n_{\pm}}{\partial \mathbf{r}^2} \right) \tag{3.13}$$

for which one can also write

$$\frac{\partial \mathbf{V}_\pm}{\partial t} + \mathbf{V}_\pm \cdot \frac{\partial \mathbf{V}_\pm}{\partial \mathbf{r}} = \mp \frac{1}{m_p} \frac{\partial U}{\partial \mathbf{r}} + \frac{\alpha \hbar^2}{2m_p^2} \frac{\partial}{\partial \mathbf{r}} \left(\frac{1}{\sqrt{n_\pm}} \frac{\partial^2 \sqrt{n_\pm}}{\partial \mathbf{r}^2} \right) \tag{3.14}$$

The equivalence of (3.14) with the Schrödinger equation (2.1) for one Planck mass can now be established by Madelung's transformation:

$$\begin{aligned} n_\pm &= \psi^* \psi \\ n_\pm \mathbf{V}_\pm &= \mp \frac{i\hbar}{2m_p} [\psi_\pm^* \nabla \psi_\pm - \psi_\pm \nabla \psi_\pm^*] \end{aligned} \tag{3.15}$$

by which (2.1) is transformed into

$$\begin{aligned} &\frac{\partial n_\pm}{\partial t} + \frac{\partial (n_\pm \mathbf{V}_\pm)}{\partial \mathbf{r}} = 0 \\ &\frac{\partial \mathbf{V}_\pm}{\partial t} + \mathbf{V}_\pm \cdot \frac{\partial \mathbf{V}_\pm}{\partial \mathbf{r}} = \mp \frac{1}{m_p} \frac{\partial}{\partial \mathbf{r}} [U + Q_\pm] \end{aligned} \tag{3.16}$$

where

$$Q_\pm = \mp \frac{\hbar^2}{2m_p} \frac{1}{\sqrt{n_\pm}} \frac{\partial^2 \sqrt{n_\pm}}{\partial \mathbf{r}^2} \tag{3.17}$$

The connection between (2.1) and (3.16) is given by

$$\begin{aligned} &\psi_\pm = A_\pm e^{iS_\pm}, \qquad A_\pm > 0, \quad 0 \le S_\pm \le 2\pi \\ &n_\pm = A_\pm^2, \qquad \mathbf{V}_\pm = \pm \frac{\hbar}{m_p} \frac{\partial S_\pm}{\partial \mathbf{r}} \end{aligned} \tag{3.18}$$

where the uniqueness of $\psi_\pm$ requires that

$$\oint \mathbf{V}_\pm \cdot d\mathbf{r} = \pm nh/m_p, \qquad n = 0, 1, 2 \ldots \tag{3.19}$$

Comparison of (3.16) with (3.14) shows complete equivalence for $\alpha = 1$, implying that $\mathbf{v}_{\mathrm{rel}} = c$.

We now show that it is even possible to obtain an expression for the collision integral taking into account higher order terms, otherwise suppressed by the Planck length. As before, assuming that $f_\pm \simeq f_\mp$, one has

$$\begin{aligned} \log f'_\pm f'_\mp &= \log f_\pm\left(\mathbf{r} \pm \frac{\mathbf{r}_p}{2}\right) + \log f_\mp\left(\mathbf{r} \mp \frac{\mathbf{r}_p}{2}\right) \\ &\simeq \log f_\pm\left(\mathbf{r} \pm \frac{\mathbf{r}_p}{2}\right) + \log f_\pm\left(\mathbf{r} \mp \frac{\mathbf{r}_p}{2}\right) \end{aligned}$$

$$= \exp\left(\pm\frac{\mathbf{r}_p}{2}\cdot\frac{\partial}{\partial\mathbf{r}}\right)[\log f_\pm] + \exp\left(\mp\frac{\mathbf{r}_p}{2}\cdot\frac{\partial}{\partial\mathbf{r}}\right)[\log f_\pm]$$

$$= 2\cosh\left(\pm\frac{\mathbf{r}_p}{2}\cdot\frac{\partial}{\partial\mathbf{r}}\right)[\log f_\pm] \tag{3.20}$$

where $\frac{1}{2}\mathbf{r}_p\cdot\partial/\partial\mathbf{r}$ is an operator for which

$$\left(\frac{\mathbf{r}_p}{2}\cdot\frac{\partial}{\partial\mathbf{r}}\right)^n = \left(\frac{\mathbf{r}_p}{2}\right)^n\cdot\frac{\partial^n}{\partial\mathbf{r}^n}$$

Hence

$$f'_\pm f'_\mp - f_\pm f_\mp \simeq \exp\left\{2\cosh\left(\pm\frac{\mathbf{r}_p}{2}\cdot\frac{\partial}{\partial\mathbf{r}}\right)[\log f_\pm]\right\} - f_\pm^2 \tag{3.21}$$

To obtain from (3.21) the approximation (3.9), we expand the hyperbolic function up to the second order and the exponential function up to first order:

$$\exp\{\cdot\} - f_\pm^2 = \exp\left\{\log f_\pm^2 + \left(\frac{\mathbf{r}_p}{2}\right)^2\frac{\partial^2\log f_\pm}{\partial\mathbf{r}^2}\right\} - f_\pm^2$$

$$\simeq \left(\frac{\mathbf{r}_p}{2}\right)^2 f_\pm^2\,\frac{\partial^2\log f_\pm}{\partial\mathbf{r}^2} \tag{3.22}$$

which is the same as (3.9).

Inserting (3.21) into (3.3), putting $\alpha = 1$, applying the averaging operator $(\mathbf{r}_p/2)\mathbf{c}\ \partial^2/\partial\mathbf{v}_\pm\partial\mathbf{r}$, and finally integrating over $d\mathbf{v}_\mp$, whereby $\int f_\mp d\mathbf{v}_\mp \simeq 1/2\mathbf{r}_p^2$, we obtain

$$\frac{\partial f_\pm}{\partial t} + \mathbf{v}_\pm\cdot\frac{\partial f_\pm}{\partial\mathbf{r}} \mp \frac{1}{m_p}\frac{\partial U}{\partial\mathbf{r}}\cdot\frac{\partial f_\pm}{\partial\mathbf{v}_\pm}$$

$$= -c^2\frac{\partial^2}{\partial\mathbf{v}_\pm\partial\mathbf{r}}\left\{f_\pm\left[\frac{\exp\{2\cosh(\pm\frac{1}{2}\mathbf{r}_p\cdot(\partial/\partial\mathbf{r})[\log f_\pm]\}}{f_\pm^2} - 1\right]\right\} \tag{3.23}$$

Integrating (3.23) over $d\mathbf{v}_\pm$, we obtain as before the continuity equation (3.11). Multiplying (3.23) by $\mathbf{v}_\pm$ and integrating over $d\mathbf{v}_\pm$, and setting $\partial^2\log f_\pm/\partial\mathbf{r}^2 \simeq \partial^2\log n_\pm/\partial\mathbf{r}^2$, we find

$$\frac{\partial(n_\pm\mathbf{V}_\pm)}{\partial t} + \frac{\partial(n_\pm\mathbf{V}_\pm\cdot\mathbf{V}_\pm)}{\partial\mathbf{r}}$$

$$= \mp\frac{n_\pm}{m_p}\frac{\partial U}{\partial\mathbf{r}} + \mathbf{c}^2\frac{\partial}{\partial\mathbf{r}}\left\{n_\pm\left[\frac{\exp\{2\cosh(\pm\frac{1}{2}\mathbf{r}_p\cdot(\partial/\partial\mathbf{r})[\log n_\pm]\}}{n_\pm^2} - 1\right]\right\} \tag{3.24}$$

which with (3.11) can be simplified:

$$\frac{\partial \mathbf{V}_\pm}{\partial t} + \mathbf{V}_\pm \cdot \frac{\partial \mathbf{V}_\pm}{\partial \mathbf{r}} = \mp \frac{1}{m_p}\frac{\partial U}{\partial \mathbf{r}} + \frac{c^2}{n_\pm}\frac{\partial}{\partial \mathbf{r}}\left\{ n_\pm \left[\frac{\exp\{2\cosh(\pm\frac{1}{2}\mathbf{r}_p \cdot (\partial/\partial\mathbf{r})[\log n_\pm]\}}{n_\pm^2} - 1 \right] \right\} \tag{3.25}$$

By applying the inverted Madelung transformation to (3.25), one can obtain higher order correction terms for the Schrödinger equation. They contain nonlinear terms suppressed by the Planck length.

4. THE SCHRÖDINGER EQUATION AS A CONSEQUENCE OF THE NEWTONIAN MECHANICS OF THE PLANCK AETHER

The solution of the Boltzmann equation gave us the Schrödinger equation for a Planck mass. This derivation suffices to set up equations (1.1) and (1.2) of the Planck aether model, but it does not prove the validity of the Schrödinger equation for any mass $m \neq m_p$.

According to the Planck aether model, all finite-rest-mass particles, except the Planck masses themselves, are excitonic quasiparticles of the Planck aether. And because these quasiparticles are held together by forces transmitted through waves which in a rest frame of the Planck aether propagate with the velocity of light, they obey Lorentz invariance as a dynamic symmetry. In this dynamic interpretation of Lorentz invariance the forces holding together the quasiparticles are balanced by the fluctuations of the Planck aether, and it is for this reason that the fluctuations have to be Lorentz invariant as well. The only Lorentz-invariant frequency spectrum is

$$f(\omega)\, d\omega = \text{const} \cdot \omega^3\, d\omega \tag{4.1}$$

which in the dynamic interpretation of Lorentz invariance is valid for $\omega << \omega_p$, where $\omega_p = c/r_p$ is the Planck frequency. Because the kinetic energy term $-(\hbar^2/2m)\nabla^2\psi$ in the Schrödinger equation for a particle of mass m implies the replacement of the classical momentum $m\mathbf{v}$ by the operator $(\hbar/i)\ \partial/\partial\mathbf{r}$, the particle has the zero-point energy $E_0 = (1/2)\hbar\omega$, where $\omega = c/\lambda$ with $\lambda = \hbar/mc$. The zero-point energy $E_0 = (1/2)\hbar\omega = mc^2/\hbar$ scales as the particle mass m, and it implies a Lorentz-invariant zero-point energy spectrum in frequency space proportional to

$$\epsilon(\omega)\, d\omega = \text{const}\cdot(1/2)\hbar\omega\cdot 4\pi\omega^2\, d\omega$$
$$= \text{const}\cdot\omega^3\, d\omega \qquad (4.2)$$

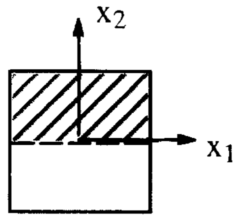
It follows that the Schrödinger equation, which we had derived from the Boltzmann equation of the Planck aether for a Planck mass $m = m_p$, remains valid for $m \neq m_p$, and we are led to the strange conclusion that it is Lorentz invariance as a dynamic symmetry which ensures the validity of the nonrelativistic Schrödinger equation for masses different from the Planck mass. However, our analysis also suggests that the Schrödinger equation becomes invalid for a mass $m >> m_p$, which should rather be described by Newtonian mechanics. If high-precision experiments could be carried out for masses $m > m_p$, they might show a departure from quantum mechanics toward classical mechanics.

The cause of the zero-point energy, which leads to the uncertainty relations, is the reason for the replacement of classical mechanics by quantum mechanics. The Planck aether hypothesis explains it through the coexistence of positive and negative masses in the Planck aether. Through them a positive-mass particle can for the time $\tau \sim \hbar/\delta E \sim \hbar/mc^2 \sim \delta/c$ (where $\delta \sim \hbar/mc$ is the Zitterbewegung displacement) "borrow" energy from the negative-mass component of the Planck aether. According to the Planck aether hypothesis, quantum mechanics has for this reason its cause in the existence of negative masses.

5. MANY-BODY SCHRÖDINGER EQUATION

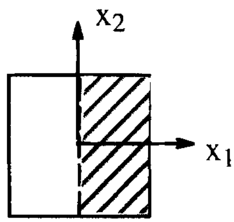
A conceptually difficult problem of quantum mechanics is the interpretation of the many-body Schrödinger equation in configuration space, because it leads to the strange phenomenon of phase entanglement. The Planck aether hypothesis can avoid this problem because it views all particles as quasiparticles of the Planck aether. According to the Planck aether hypothesis, it is incorrect to visualize a many-body wave function to be composed of the same particles which are observed before an interaction between the particles is turned on. In the Planck aether hypothesis, where all particles are quasiparticles, the interaction rather leads to a new set of quasiparticles, the wave function of which can be factorized. This can be demonstrated for the wave function of two identical particles moving in a harmonic oscillator well. The well shall have its coordinate origin at $x = 0$, with the first particle having the coordinate x_1 and the second one the coordinate x_2. Considering two oscillator wave functions $\psi_0(x)$ and $\psi_1(x)$, with ψ_0 having no and $\psi_1(x)$ having one node, there are two two-particle wave functions:

$$\psi(x_1, x_2) = \psi_0(x_1)\psi_1(x_2) = (2/\pi)^{1/2} x_2 \exp[-(x_1^2 + x_2^2)/2] =$$



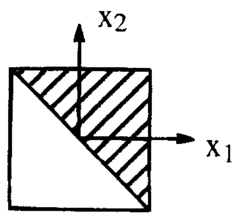
(5.1)

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_0(x_2) = (2/\pi)^{1/2} x_1 \exp[-(x_1^2 + x_2^2)/2] =$$



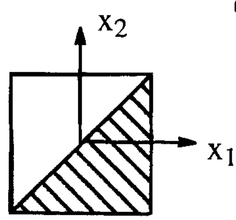
graphically displayed in the x_1, x_2 configuration space, with the nodes along the lines $x_2 = 0$ and $x_1 = 0$. By a linear superposition of these wave functions we get a symmetric and an antisymmetric combination:

$$\psi_s(x_1, x_2) = (1/\sqrt{2})[\psi_0(x_1)\psi_1(x_2) + \psi_1(x_1)\psi_0(x_2)] = (1/\sqrt{\pi})(x_2 + x_1) \exp[-(x_1^2 + x_2^2)/2] =$$



(5.2)

$$\psi_a(x_1, x_2) = (1/\sqrt{2})[\psi_0(x_1)\psi_1(x_2) - \psi_1(x_1)\psi_0(x_2)] = (1/\sqrt{\pi})(x_2 - x_1) \exp[-(x_1^2 + x_2^2)/2] =$$



If a perturbation is applied whereby the two particles slightly attract each other, the degeneracy for the two wave functions is removed, with the symmetric wave function leading to a lower energy eigenvalue. For a repulsive force between the particles the reverse is true. As regards the wave functions (5.1), one may still think of them in terms of two particles, because the wave functions can be factorized, with the quantum potential becoming a sum of two independent terms

$$-\frac{\hbar^2}{2m}\frac{\nabla^2(\psi^*\psi)^{1/2}}{(\psi^*\psi)^{1/2}}$$
$$= -\frac{\hbar^2}{2m}\frac{1}{(\psi_1^*\psi_1)^{1/2}}\frac{\partial^2(\psi_1^*\psi_1)^{1/2}}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{1}{(\psi_2^*\psi_2)^{1/2}}\frac{\partial^2(\psi_2^*\psi_2)^{1/2}}{\partial x_2^2} \tag{5.3}$$

Such a decomposition into parts is not possible for the wave functions (5.2), and it is then no longer possible to think of the two particles which are placed into the well. This, however, is possible by making a 45° rotation in configuration space. Putting

$$y = x_2 + x_1$$
$$x = x_2 - x_1 \tag{5.4}$$

one obtains the factorized wave functions

$$\psi_s = \frac{1}{\sqrt{\pi}} y e^{-(x^2+y^2)/2}$$
$$\psi_a = \frac{1}{\sqrt{\pi}} x e^{-(x^2+y^2)/2} \tag{5.5}$$

for which the quantum potential separates into a sum of two independent terms, one depending only on x and the other one only on y. This means that the addition of a small perturbation in the form of an attraction or repulsion between the two particles transforms them into a new set of two quasiparticles different from the original particles.

With the identification of all particles as quasiparticles of the Planck aether, the abstract notion of configuration space and inseparability into parts disappears, because any many-body system can, in principle, at each point always be expressed as a factorizable wave function of quasiparticles, where the quasiparticle configuration may change from point to point. This can be shown quite generally. For an N-body system, the potential energy can at each point of configuration space be expanded into a Taylor series

$$U = \sum_{k,l}^{N} a_{kl} x_k x_l \tag{5.6}$$

Together with the kinetic energy

$$T = \sum_{l}^{N} \frac{m_l}{2} \dot{x}_l^2 \tag{5.7}$$

one obtains the Hamilton function $H = T + U$ and from there the many-body Schrödinger equation. Introducing the variables $\sqrt{m_l} x_l = y_l$, one has

$$T = \sum_{l}^{N} \frac{1}{2} \dot{y}_l^2$$

$$U = \sum_{k,l}^{N} b_{kl} y_k y_l \tag{5.8}$$

which by a principal axis transformation of U become

$$T = \sum_{l} \frac{1}{2} \dot{z}_l^2$$

$$U = \sum_{l} \frac{\omega_l^2}{2} z_l^2 \tag{5.9}$$

Unlike the Schrödinger equation for the potential (5.6), the potential (5.9) leads to a completely factorizable wave function, with a sum of quantum potentials each depending only on one quasiparticle coordinate. The transformation from (5.8) to (5.9) is used in classical mechanics to obtain the normal modes for a system of coupled oscillators. The quasiparticles into which the many-body wave function can be factorized are then simply the quantized normal modes of the corresponding classical system.

For the particular example of two particles placed in a harmonic oscillator well, the normal modes of the classical mechanical system are those where the particles either move in phase or out of phase by 180°. In quantum mechanics, the first mode corresponds to the symmetric, the second one to the antisymmetric wave function. It is clear that the quasiparticles representing the symmetric and antisymmetric modes cannot be localized at the position of the particles placed into the well.

6. WAVE FUNCTION COLLAPSE

As von Neumann has shown, quantum mechanics consists of two quite different procedures: (1) a deterministic evolution of the wave function by Schrödinger's equation, and (2) an indeterministic process whereby through a measurement the wave function "collapses" with superluminal speed into one of many alternatives, with the probability for one of the alternatives actually to occur expressed by the wave function prior to the measurement.

In the Copenhagen interpretation, the wave function has no real physical meaning, being rather the expression of our knowledge. As our knowledge can change discontinuously following a measurement, so can the wave function. It is for this reason that the collapse of the wave function can occur with superluminal speed. Even though a measurement can always be carried out by an instrument, the Copenhagen interpretation ultimately requires the existence of conscious observers, introducing a highly subjective element into the description of nature. With few places in the physical universe having conscious observers present, the Copenhagen interpretation has not been accepted by all physicists.

In a Newtonian interpretation of quantum mechanics, not only would the Schrödinger equation have to be mechanistically derived, but superluminal wave function collapse as well. One may wonder if superluminal wave function collapse might not be in violation of special relativity, but there are two reasons why this is really not the case. First, as Ehrenfest (1927) has shown, a wave packet under the influence of an external force behaves like a particle in classical mechanics. Accordingly, as long as the center of mass of the wave packet does not assume superluminal velocities, there is no reason against an internal superluminal motion within the wave packet. But it is only this kind of superluminal motion which is required for wave function collapse. Second, because the Planck aether has all the characteristics of a medium, it can have wave modes with divergent phase velocities. As long as the superluminal collapse velocity does not transmit a signal, there can be no violation of special relativity.

To show how superluminal wave function collapse may perhaps be understood as a mechanical effect of the Planck aether, we may choose the Hartree approximation, the simplest approximation of (1.1), whereby the field operators $\psi_\pm$ are replaced by their expectation values $\phi_\pm = \langle\psi_\pm\rangle$ and $\phi^*_\pm = \langle\psi^\dagger_\pm\rangle$, yielding the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\phi_\pm}{\partial t} = \mp\frac{\hbar^2}{2m_p}\nabla^2\phi_\pm \pm 2\hbar c r_p^2(\phi^*_\pm\phi_\pm - \phi^*_\mp\phi_\mp)\phi_\pm \tag{6.1}$$

By the Madelung transformation it becomes

$$\begin{aligned}\frac{\partial\mathbf{V}_\pm}{\partial t} + \nabla\left(\frac{\mathbf{V}_\pm^2}{2}\right) &= -2c^2r_p^3\nabla(n_\pm - n_\mp) + \frac{1}{m_p}\nabla Q_\pm \\ \frac{\partial n_\pm}{\partial t} + \nabla\cdot(n_\pm\mathbf{V}_\pm) &= 0\end{aligned} \tag{6.2}$$

where

$$Q_\pm = \frac{\hbar^2}{2m_p}\frac{\nabla^2\sqrt{n_\pm}}{\sqrt{n_\pm}} \tag{6.3}$$

is the quantum potential. For small-amplitude disturbances with wavelengths large compared to the Planck length, one can neglect the quantum potential, and obtains from (6.2)

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\mathbf{V}_+ + \mathbf{V}_-) = 0 \\
&\frac{\partial}{\partial t}(\mathbf{V}_+ - \mathbf{V}_-) = -4c^2 r_\mathrm{p}^3 \nabla(n'_+ - n'_-) \\
&\frac{\partial n'_\pm}{\partial t} + \frac{1}{2r_\mathrm{p}^3}\nabla\cdot\mathbf{V}_\pm = 0
\end{aligned}
\tag{6.4}
$$

where $n_\pm = 1/2r_\mathrm{p}^3 + n'_\pm$. Eliminating $n'_\pm$ from the second and third relations of (6.4), one obtains the wave equation

$$\frac{\partial^2}{\partial t^2}(\mathbf{V}_+ - \mathbf{V}_-) = 2c^2\nabla^2(\mathbf{V}_+ - \mathbf{V}_-) \tag{6.5}$$

with the dispersion relation

$$\omega^2 = 2c^2k^2 \tag{6.6}$$

For oscillatory disturbances the first relation of (6.4) implies that $\mathbf{V}_- = -\mathbf{V}_+$ and hence $n'_- = -n'_+$, whereby the total number density of the positive and negative Planck masses remains unchanged. Accordingly, the wave does not carry any energy and is "empty".

Next we must consider the coupling of these disturbances with a particle described by the Schrödinger wave function. We first consider the interaction with the Schrödinger wave for a Planck mass. To be described by a Schrödinger equation , it must be distinct from the Planck masses of the Planck aether. This is true for a Planck mass bound in a quantized vortex filament, with the diameter of the filament equal to a Planck length. Being bound in the vortex filament, the Planck mass executes zero-point oscillations determined by the uncertainty principle. This zero-point energy is $\hbar c/r_\mathrm{p}$ and it generates a virtual phonon field surrounding the Planck mass with the strength of this field equal to the strength of the scalar Newtonian gravitational field of a Planck mass. Unlike the better Hartree–Fock approximation, the Hartree approximation does not lead to quantized vortex solutions in the positive-negative-mass Planck aether, but the Hartree approximation has the phonon–roton spectrum of a superfluid, and a Planck mass bound in a roton would qualitatively behave like one bound in a vortex filament. Therefore, to make the analysis as simple as possible, we can use the Hartree approximation.

Assuming that all Planck masses belonging to the disturbances $n'_\pm$ are bound in rotons, (6.2) gives the following set of small-amplitude equations, with the Planck masses bound in rotons generating a scalar gravitational potential Φ:

$$\frac{\partial \mathbf{V}_\pm}{\partial t} = -2c^2 r_p^3 \nabla(n'_\pm - n'_\mp) - \nabla\Phi$$

$$\frac{\partial n'_\pm}{\partial t} + \frac{1}{2r_p^3}\nabla\cdot\mathbf{V}_\pm = 0$$

$$\nabla^2\Phi = 4\pi G m_p(n'_+ - n'_-) \tag{6.7}$$

where as before we have neglected the quantum potential. With $Gm_p^2 = \hbar c$, and the second relation of (6.7), one obtains for the gravitational potential Φ

$$\nabla^2 \frac{\partial\Phi}{\partial t} = -2\pi\omega_p^2\nabla\cdot(\mathbf{V}_+ - \mathbf{V}_-) \tag{6.8}$$

From (6.7) and (6.8) one then obtains

$$\frac{\partial^2}{\partial t^2}(\mathbf{V}_+ + \mathbf{V}_-) = 4\pi\omega_p^2(\mathbf{V}_+ - \mathbf{V}_-)$$

$$\frac{\partial^2}{\partial t^2}(\mathbf{V}_+ - \mathbf{V}_-) = 2c^2\nabla^2(\mathbf{V}_+ - \mathbf{V}_-) \tag{6.9}$$

As before, the second relation of (6.9) has wavelike disturbances obeying the dispersion relation (6.6), but it has in addition also the special solution $\mathbf{V}_+ - \mathbf{V}_- = A =$ const. Inserting this special solution into the first relation of (6.9), one obtains for $(\mathbf{V}_+ + \mathbf{V}_-)$ a solution rising in time:

$$(\mathbf{V}_+ + \mathbf{V}_-) = 2\pi\omega_p^2(\mathbf{V}_+ - \mathbf{V}_-)t^2 \tag{6.10}$$

For $\mathbf{V}_- = 0$, with $\omega_p^2 = {}^2Gn_+m_p = {}^2G\rho$, (6.10) becomes

$$t = 1/(4\pi G\rho)^{1/2} \tag{6.11}$$

which is the gravitational collapse time for a mass of density ρ. If $\mathbf{V}_- \to -\mathbf{V}_+$, by which the second relation of (6.9) approaches the "empty" wave solution, one has $t \to 0$. The gravitational collapse time can for this reason be substantially shortened in the presence of negative masses if the negative mass flow is in a direction opposite to the flow of the positive masses. Because the shortening of the collapse time occurs when the net average density approaches zero, as is the case for the "empty" wave, we suggest that this kind of gravitational collapse may serve as a model for wave function collapse.

Assuming that the ratio $\hbar\omega/\hbar\omega_p$ of the kinetic energy of the Planck mass described by Schrödinger's equation to the Planck energy is equal to $(|\mathbf{V}_+|^2$

$- |\mathbf{V}_-|^2)/(|\mathbf{V}_+|^2 + |\mathbf{V}_-|^2)$, we can set near $\mathbf{V}_- \simeq -\mathbf{V}_+$, $\omega/\omega_p \simeq (|\mathbf{V}_+| + |\mathbf{V}_-|)/|\mathbf{V}_+|$, hence $(\mathbf{V}_+ + \mathbf{V}_-)/(\mathbf{V}_+ - \mathbf{V}_-) = \frac{1}{2}\omega/\omega_p$. We thus find for (6.10)

$$t^2 = \frac{\omega}{4\pi\omega_p^3} \tag{6.12}$$

Because the time for the collapse should be of the order $t \sim 1/\omega$, one finally has

$$t \simeq (4\pi)^{-1/3} r_p/c \tag{6.13}$$

A wave packet of width r of a Planck mass described by Schrödinger's equation would collapse with the superluminal speed

$$v_c = r/t \sim (r/r_p)c \tag{6.14}$$

In generalizing this result to a Schrödinger equation describing a mass $m < m_p$, we have to replace in (6.4) and (6.7) r_p with $r_0 = \hbar/mc$, and find instead of (6.13)

$$t \simeq (4\pi)^{-1/3} r_0/c \tag{6.15}$$

with (6.5) remaining unchanged. For the collapse velocity we obtain instead of (6.14)

$$v_c \sim (r/r_0)c \tag{6.16}$$

For the collapse to proceed along the lines suggested by the model, the wavelike disturbances of the Planck aether must be in phase. With the Planck aether likely to be subject to large-scale fluctuations, possibly rising in proportion to $r^{1/3}$ as for a turbulent fluid, the mechanism for the collapse may not work above a certain length. If this should turn out to be true, then the quantum mechanical correlations are going to break down above this length.

7. DISCUSSION

We have shown that the laws of quantum mechanics can be derived from Newtonian mechanics of the Planck aether. We have also shown that the Planck aether hypothesis removes the mystery of nonfactorizable many-body wave functions in configuration space, because in the Planck aether hypothesis all particles are quasiparticles of the Planck aether which through an interaction may continuously change into a new set of quasiparticles with factorizable wave functions. The remaining problem for a completely classical mechanical interpretation of quantum mechanics is the collapse of the wave function. The Copenhagen interpretation avoids addressing this problem, but at a very high price, by demanding the presence of conscious observers. The presence of negative masses in the Planck aether leads to a kind of gravitational collapse which might explain the phenomenon of superluminal wave function collapse.

Finally, we would like to compare our theory with past attempts to find a classical physics explanation of quantum mechanics. These attempts were pioneered by Bohm (1952), who showed how a simple model could explain the one-body Schrödinger equation by assigning the Schrödinger wave function the property of a classical field guiding the particle along a trajectory disturbed by a Zitterbewegung of unknown origin. The next step was taken by Fényes (1952) who postulated the existence of a Zitterbewegung of unknown origin given by the diffusion velocity

$$\mathbf{v}_D = -\frac{\hbar}{2m}\left(\frac{\nabla n}{n}\right) \tag{7.1}$$

He then showed that the Schrödinger equation can be derived from the variational principle

$$\delta \int n\left\{\hbar \frac{\partial S}{\partial t} + \frac{\hbar^2}{2m}(\nabla S)^2 + U + \frac{m}{2}\left(\frac{\hbar}{2m}\frac{\nabla n}{n}\right)^2\right\} d\mathbf{r} = 0 \tag{7.2}$$

where n and S are given by (3.15) and (3.18). A different proposal made by Weizel (1953a,b, 1954) postulated as the cause for the Zitterbewegung hypothetical particles called "zerons". A somewhat different version of Weizel's idea was presented by Nelson (1966). All these proposals were criticized by Heisenberg (1963). Bohm's idea was criticized for failing to provide an understanding of the many-body Schrödinger wave in configuration space. Fényes was criticized for the lack of a physical interpretation of the statistical laws he proposed. Weizel's model, finally, was criticized because it seemed to be in violation of the second law of thermodynamics, with the zerons gaining in entropy in the course of the diffusion process. In the Planck aether hypothesis, where the diffusion is caused by negative-Planck-mass particles, and where in the course of the collisions the particles do not change their velocity, the entropy does not change. Furthermore, since in the Planck aether hypothesis all particles are quasiparticles of the Planck aether, with multiparticle configurations expressed in terms of factorizable quasiparticle wave functions, Heisenberg's other criticism raised against Bohm's proposal does not apply either.

This brings us to the most difficult problem, superluminal wave function collapse. An interesting idea toward an explanation has been made by Penrose (1989), who gives strong reasons why wave function collapse might be related to gravitational collapse. However, it is difficult to see how gravitational collapse alone can explain wave function collapse in microphysics in the absence of any other mechanism. The one "graviton" criterion Penrose proposes as a heuristic model remains unconvincing to this author. This problem does not arise in the Planck aether hypothesis, where gravitational collapse is greatly enhanced through the existence of negative masses.

In the Planck aether hypothesis the negative masses play the role of "hidden parameters", hidden indeed, because it would require an energy comparable to the Planck energy of $\sim 10^{19}$ GeV to make them directly observable.

REFERENCES

Bohm, D. (1952). *Physical Review,* **85**, 166, 180.
Fényes, I. (1952). *Zeitschrift für Physik,* **132**, 81.
Ehrenfest, P. (1927). *Zeitschrift für Physik,* **45**, 455.
Heisenberg, W. (1963). *Physik and Philosophie,* Ullstein, pp. 104ff.
Landau, L. D., and Lifshitz, E. M. (1981). *Course of Theoretical Physics,* Vol. 10, Pergamon Press, Oxford, pp. 7ff.
Nelson, E. (1966). *Physical Review,* **150**, 1079.
Penrose, R. (1989). *The Emperor's New Mind,* Oxford University Press, Oxford, pp. 348ff.
Schrödinger, E. (1930). *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse,* **1930**, 416.
Schrödinger, E. (1931). *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse,* **1931**, 418.
Weizel, W. (1953a). *Zeitschrift für Physik,* **134**, 264.
Weizel, W. (1953b). *Zeitschrift für Physik,* **135**, 270.
Weizel, W. (1954). *Zeitschrift für Physik,* **136**, 582.
Winterberg, F. (1988). *Zeitschrift für Naturforschung,* **43a**, 1131.
Winterberg, F. (1994). *International Journal of Theoretical Physics,* **33**, 1275.